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THE STABILITY OF PERIODIC TRAJECTORIES OF A BILLIARD BALL IN THREE DIMENSIONS†

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The stability of periodic trajectories of a material point moving between two convex walls with elastic reflections is investigated. The problem is closely bound related to wave propagation theory in the shortwave approximation [1]. The simplest periodic trajectory is a section of a straight line orthogonal to the walls at its endpoints. The problem of the stability of a two-part trajectory was solved in [1] in two dimensions. It will be solved here in three dimensions using the method developed in [2].

1. STATEMENT OF THE PROBLEM

WE WISH to investigate the stability of a two-part periodic trajectory of a Birkhoff billiard ball with three degrees of freedom (Fig. 1); a material point is moving constantly along a section of length l, being periodically reflected elastically from a boundary surface Σ. Let O_1 and O_2 be the endpoints of the two-part trajectory. We introduce Cartesian coordinates x, y, z, with the z axis lying along O_1O_2. In the neighbourhood of O_1, O_2 the equation of the surface Σ has the form

$$z = f_1(x, y) = a_1x^2 + b_1xy + c_1y^2 + o(x^2 + y^2)$$
$$z = f_2(x, y) = l - a_2x^2 - b_2xy - c_2y^2 + o(x^2 + y^2)$$

We can take as the parameter of the periodic motion the velocity v of the point. Following [2], we replace the unilateral constraints of the elastic force field by potential

$$V_N = \begin{cases} 0, & f_1 \leqslant z \leqslant f_2 \\ N(z - f_i)^2/2, & z \leqslant f_1 \text{ and } z \geqslant f_2; \quad i = 1, 2 \end{cases}$$

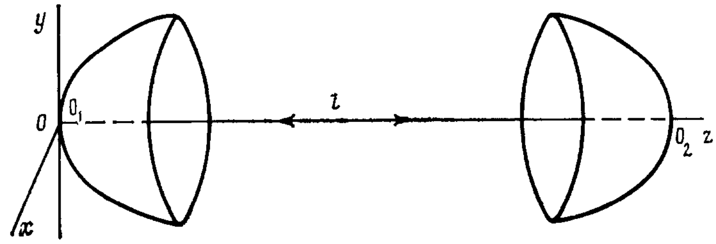

FIG. 1.

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Here N is a positive parameter, which will subsequently approach $+\infty$. The problem of the motion of a point m in a field with potential V_N has a family of $2T$-periodic solutions:

$$x_0(t) \equiv 0, \; y_0(t) \equiv 0, \; 0 \leqslant t \leqslant 2T$$
$$z_0(t) = \begin{cases} vN^{-1/2} \sin N^{-1/2} t, & 0 \leqslant t \leqslant \tau \\ v(t-\tau), & \tau \leqslant t \leqslant T \\ vN^{-1/2} \sin N^{-1/2}(t+T), & T \leqslant t \leqslant T+\tau \\ v(t-T-\tau), & T+\tau \leqslant t \leqslant 2T \end{cases} \tag{1.1}$$
$$\tau = \pi N^{-1/2}, \; T = \tau + l/v$$

Using Lyapunov's method [3], we will now investigate the stability of the periodic motion (1.1) in the linear approximation, assuming that N is sufficiently large. As shown in [2], as $N \to +\infty$ the conditions for the solution (1.1) to be stable become the conditions for the stability of a two-part trajectory of a Birkhoff billiard ball.

2. THE GENERALIZED LYAPUNOV METHOD

We will write the variational equations for the periodic solution (1.1) as a linear system of first-order differential equations:

$$(\delta z)^{\cdot} = A(t)\,\delta z, \quad A = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -2a_i p_i(t) & 0 & -b_i p_i(t) & 0 \\ 0 & 0 & 0 & 1 \\ -b_i p_i(t) & 0 & -2c_i p_i(t) & 0 \end{Vmatrix} \tag{2.1}$$

where $p_i(t) = (-1)^i vN^{-1/2} \sin N^{-1/2} t$ if $t \in (0, \tau)$, $i = 1$, and $t \in (T, T+\tau)$, $i = 1$, $p_i(t) \equiv 0$ for other values of t.

Let $X(t)$ be a fundamental matrix for systems (2.1): $X^{\cdot} = AX$, $X(0) = E$. The multipliers of the periodic solution (1.1) are the roots of the characteristic equation $|X(2T) - \rho E| = 0$. Since the motion of the point in a field with potential V_N is described by the Hamilton equations, the characteristic equation is reciprocal [3]:

$$\rho^4 + a\rho^3 + b\rho^2 + a\rho + 1 = 0, \; a, \; b \in \mathbf{R} \tag{2.2}$$

The left-hand side of (2.2) may be resolved into factors:

$$(\rho^2 + d_+\rho + 1)(\rho^2 + d_-\rho + 1) = 0$$
$$d_\pm = [a \pm (a^2 - 4b + 8)^{1/2}]/2 \tag{2.3}$$

In the general case the coefficients $d_\pm$ may be complex numbers.

A necessary condition for the solution (1.1) to be stable is that all the roots of Eq. (2.2) should lie on the unit circle. This condition is satisfied if and only if $d_\pm$ are real and $|d_\pm| \leqslant 2$. Together with (2.3) this gives a system of inequalities defining the relationship between the real coefficients a, b:

$$|a| \leqslant 4, \; 2a - b \leqslant 2, \; 2a + b \geqslant -2, \; b \leqslant a^2/4 + 2 \tag{2.4}$$

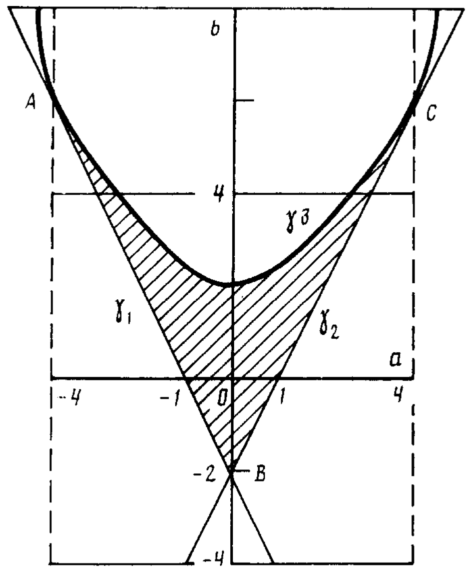

FIG. 2.

The domain thus defined in the a, b plane has a piecewise-smooth boundary, which is the union of four curves γ_i (Fig. 2).

If (a, b) is a point of the boundary segment γ_1, the periodic motion is degenerate: one pair of multipliers coincides with the point $\rho = 1$, the other pair lies on the unit circle. If $(a, b) \in \gamma_2$, one pair of multipliers coincides with the point $\rho = -1$, the other is again on the unit circle. Finally, if $(a, b) \in \gamma_3$, the two pairs of complex-conjugate multipliers are identical. The corner points A, B in Fig. 2 correspond to the cases in which $\rho = 1$ and $\rho = -1$ are four-fold roots of the characteristic equation (2.2); the point C corresponds to double roots $\rho = 1$ and $\rho = -1$.

To determine the coefficients a, b we use Lyapunov's method (see [3]). To that end we introduce a parameter μ in Eq. (2.1):

$$(\delta z)^{\cdot} = \mu A\,(t)\,\delta z$$

Then the solutions can be expressed as series in powers of μ, which converge for all $|\mu| \leqslant 1$. Now, putting $\mu = 1$, we obtain expressions for the monodromy matrix elements, as convergent series. Now let the parameter N go to infinity. It turns out that in the limit the series become finite sums, so that explicit stability conditions can be derived (see [2]). Omitting the details, we present the final formulae for the coefficients a and b:

$$\begin{gathered}
a = -4 + 8\sigma l - (h_a + h_c)\,l^2, \quad b = 6 + k_1 l + k_2 l^2 + k_3 l^3 + k_4 l^4 \\
k_1 = -16\sigma, \quad k_2 = h_a + h_c + 64\,(a_1 + a_2)\,(c_1 + c_2) + \\
+ 16\,(a_1 a_2 + c_1 c_2) - 16 b_1^2 - 24 b_1 b_2 - 16 b_2^2 \\
k_3 = -8\,(c_1 + c_2)\,h_a - 8\,(a_1 + a_2)\,h_c + 8\,(b_1 + b_2)\,H \\
k_4 = h_a h_c - H^2, \quad \sigma = a_1 + a_2 + c_1 + c_2 \\
h_q = 4\,[(4q_1^2 + b_1^2)\,(4q_2^2 + b_2^2)]^{1/2}, \quad q = a,\ c \\
H = 8\,[b_1 b_2\,(a_1 + c_1)\,(a_2 + c_2)]^{1/2}
\end{gathered} \tag{2.5}$$

3. STABILITY CONDITIONS

Let us analyse inequalities (2.4) together with (2.5). We will first consider the special case in which $b_1 = b_2 = 0$. Assuming that the surface Σ is convex and also, to fix our ideas, that $0 < a_1 \leqslant a_2$, $0 < c_1 \leqslant c_2$, we obtain the following stability conditions in the linear approximation from (2.4):

$$\begin{aligned} &l \leqslant {}^1/_2 a_2 \quad \text{or} \quad {}^1/_2 a_1 \leqslant l \leqslant {}^1/_2 a_1 + {}^1/_2 a_2 \\ &l \leqslant {}^1/_2 c_2 \quad \text{or} \quad {}^1/_2 c_1 \leqslant l \leqslant {}^1/_2 c_1 + {}^1/_2 c_2 \end{aligned} \tag{3.1}$$

In that case Eqs (2.1) split into two independent Hill equations and inequalities (3.1) are the stability conditions for a two-part periodic trajectory of a Birkhoff billiard ball in two dimensions (when $y \equiv 0$ or $x \equiv 0$, respectively) [1].

We will now analyse the general case. A suitable rotation of the x and y axes will make the coefficient b_1 vanish.

The last inequality in system (2.4) is always true. Indeed, it can be expressed in the following equivalent form:

$$\begin{aligned} &[l\,(\Lambda_c + \Lambda_c) - (a_1 + a_2 - c_1 - c_2)]^2 + {}^1/_2\,(\Lambda_a - 2a_1a_2) + \\ &+ {}^1/_2\,(\Lambda_c - 2c_1c_2) + b_2^2 \geqslant 0, \quad \Lambda_q = q_1\,(4q_2^2 + b_2^2)^{1/2}, \quad q = a,\ c \end{aligned} \tag{3.2}$$

This becomes an equality only when $b_2 = 0$, but then the stability conditions reduce to (3.1). If $b_2 \neq 0$, then (3.2) is a strict inequality and therefore the pairs of complex-conjugate multipliers are distinct (see Sec. 2).

Thus, the stability conditions in the typical case ($b_2 \neq 0$) are

$$|\,a\,| \leqslant 4, \quad 2a + b \geqslant -2, \quad 2a - b \leqslant 2$$

where a and b are determined from (2.5). Now, fixing values of the parameters a_1, b, c_1, let us vary the distance l between the walls from zero to infinity. We wish to investigate the behaviour of the multipliers in the complex plane as l is increased.

If $l \to 0$ then, by (2.5), $a \to -4$ and $b \to 6$. Now increase l. Relations (2.5) define a certain curve in the plane $\mathbf{R}^2 = \{a, b\}$, the form of which is an indicator of the stability of the two-part trajectory. If $\sigma < 0$ then, by (2.5), $a < -4$ for all $l > 0$ and therefore the periodic trajectory under consideration is unstable. We shall therefore assume henceforth that $\sigma > 0$. Then as l is increased a increases and reaches its maximum at

$$l = l_a = \sigma/[2\,(\Lambda_a + \Lambda_c)] \tag{3.3}$$

For small l values the point $(a(l),\ b(l))$ lies within the hatched domain in Fig. 2, provided that $2a(l) + b(l) > -2$. This is equivalent to the condition

$$\Delta \geqslant 0, \quad \Delta = 8\,(a_1 + a_2)\,(c_1 + c_2) + 2\,(a_1a_2 + c_1c_2) - 2b_2^2 - \Lambda_a - \Lambda_c \tag{3.4}$$

We will show that this inequality will certainly hold if the surface Σ is convex (when $a_1, c_1 > 0$ and $4a_2c_2 \geqslant b_2^2$). Indeed, we use the inequalities

$$(4a_2^2 + b_2^2)^{1/2} \leqslant 2a_2 + c_2, \quad (4c_2^2 + b_2^2)^{1/2} \leqslant 2c_2 + a_2$$

Then the left-hand side of inequality (3.4) is not less than the sum $8a_1a_1 + 7(a_2c_1 + a_1c_2)$, which is always positive. Thus, if Σ is convex the two-part trajectory is stable for small l values.

Now increase l. Then (a, b) will lie on the straight line that includes γ_1, provided that l is a root of the quadratic equation

$$8\Lambda_a\Lambda_c l^2 - 8l\,[(c_1 + c_2)\,\Lambda_a + (a_1 + a_2)\,\Lambda_c] + \Delta = 0 \tag{3.5}$$

It can be shown that Eq. (3.5) always has two real roots; moreover, if inequality (3.4) is true both roots are positive; otherwise, one of them is negative. In addition, if l is the greater of the positive roots, then $a(l) < -4$.

Thus, if the point $(a(l),\ b(l))$, beginning at the corner point A, enters the half-plane $2a+b<-2$, then for all $l>0$ the point (a, b) will remain outside the hatched domain in Fig. 2. We are here using continuity arguments and the fact that the curve (2.5) never cuts the parabola $b = a^2/4+2$ if $b_2 \neq 0$. Thus, inequality (3.4) is a necessary condition for the two-part trajectory to be stable.

Once in the hatched domain, the point $(a(l),\ b(l))$ can reach the boundary γ_2 provided that $2a-b=2$ or, in explicit form:

$$4\Lambda_a\Lambda_c l^4 - 4l^3\left((c_1+c_2)\Lambda_a + (a_1+a_2)\Lambda_c\right) + l\ (\Delta/2 + 2\Lambda_a + 2\Lambda_c) - - 2\sigma l + 1 = 0 \tag{3.6}$$

There are three possibilities: Eq. (3.6) has four positive roots (counting multiplicities), two positive roots or no roots $l>0$ at all. The first case will certainly hold at some time as $b_2 \to 0$. If Eq. (3.6) has no positive roots, then if $l>l_1$ [l_1 being the least of the positive roots of (3.5)] the point (a, b) will leave the hatched domain, moving subsequently in the half-plane $2a+b<-2$, which it will leave from outside the vertical strip $|a| \leq 4$.

Suppose now that Eq. (3.6) has two positive roots l_2 and l_3, where $l_2<l_3$. Then the point $(a(l),\ b(l))$ will reach γ_2 and leave the hatched domain for the half-plane $2a-b>2$, provided that $l_2<l_1$. Otherwise it will cross the straight line $2a-b=2$ outside γ_2.

If Eq. (3.6) has four positive roots $l_2<l_3<l_4<l_5$, then, depending on the value of l_1, the point $(a(l),\ b(l))$ will first cross one of the straight lines $2a-b=2$ or $2a+b=-2$. As $l>l_5$ increases further, (a, b) will cross the line $2a+b=-2$ outside the half-plane $a>-4$ and remain, for sufficiently large l, within the domain $2a+b>-2$, $a^2-4b+8>0$.

As yet another application of the above stability conditions, let us consider the case in which the convex boundary Σ is the union of two identical cylindrical surfaces, rotated relative to one another through an angle $\varphi (0 \leq \varphi \leq \pi/2)$. Then we must consider Eqs (2.6) with

$$a_1 = a > 0,\ b_1 = c_1 = 0;\ a_2 = a\cos^2\varphi,\ b_2 = a\sin 2\varphi,\ c_2 = a\sin^2\varphi$$

If $\varphi = 0$, the two-part trajectory will be degenerate: one pair of multipliers equals unity, while the other lies on the unit circle, provided that $\xi = al \leq 1$ [compare with (3.1)]. For $\xi = 1/2$, $\xi = 1$ we obtain a double multiplier $\rho = -1$, $\rho = 1$. If $\varphi = \pi/2$, both pairs of multipliers lie on the circle $|\rho| = 1$, with $\xi = 1/2$ corresponding to a pair of multipliers equal to -1. The necessary conditions for stability in the intermediate cases reduce to the following two inequalities:

$$\xi < (4 - \cos\varphi - 3\cos^2\varphi)\,(8\sin^2\varphi\cos\varphi)^{-1}$$
$$8\sin^2\varphi\cos\varphi\,\xi^3 - (4 + 3\cos\varphi - 3\cos^2\varphi)\,\xi^2 + 4\xi - 1 < 0$$

It follows from these inequalities that at small angles of rotation φ the stability conditions in the linear approximation reduce to a single inequality, $\xi < 7/16$.

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